Minimum energy asteroid reconfigurations and catastrophic disruptions[1]

Daniel J. Scheeres

ECOT 611, 429 UCB
University of Colorado at Boulder
Boulder, Colorado 80309-0429, USA

Abstract

Dramatic alteration of an asteroid's morphology need not involve high energy impacts between bodies. Simple sunlight shining on an asteroid can, through the YORP effect, cause it to undergo dramatic reconfigurations, fission into a binary asteroid or, in some cases, even undergo a catastrophic disruption with the asteroid losing a large fraction of its initial mass. This paper discusses the system level constraints and conditions for these reconfigurations to occur as a body's spin rate changes.

1 Introduction

This paper generalizes the conditions for reconfiguration and fission of a rubble pile composed of rigid bodies resting on each other and discusses theoretical limits for when such transitions occur. Given a clear understanding of these transitions, it becomes possible to map out the evolution of a rubble pile as its spin rate changes due to the YORP effect [9] or to a planetary flyby [10]. We will focus on the effect of an increasing spin rate, but some of the results we present also apply to a body subjected to a decreasing spin rate. The theory described is intended to provide a context within which more general numerical analysis of this problem can be formulated.

Our physical model focuses on a rubble pile asteroid consisting of a collection of rigid bodies resting on each other. Interesting questions to ask regarding such rubble pile asteroids include at what spin rate will the body undergo a reconfiguration of its components, at what spin rate will fission occur, and what will the dynamical outcome of fission be. Our analysis is

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distinct from previous research that has applied continuum mechanics models based on spatial averaging to these problems [5, 6]. When discrete monolithic blocks are modeled the necessary methods of analysis become different.

The main conclusions can be summarized as follows. As a given rubble pile is subject to an increasing spin rate it may pass threshold spin rates where components of the body are induced to tip or slide relative to each other. Such changes can be isolated to single blocks tipping over or may precipitate a global "landslide" on the body. If the spin rate continues to increase the asteroid can fission into multiple components in orbit about each other. For a given collection of rigid bodies, the two sub-sets that have the largest separation between their centers of mass will fission at the lowest spin rate. Thus the detailed morphology, shape, and block distribution of an asteroid are important factors in deciding its fission rate. Once fission of the asteroid occurs, its fate is strongly influenced by the free energy of the resulting system. If the free energy is positive, the system will likely disrupt and the two components will escape from each other – by definition this would constitute a catastrophic disruption of the system. If the free energy is negative the two largest components are bound to each other and will either reimpact, if sufficient angular momentum can be shed by the loss of smaller particles or by perturbations from the sun, or will enter into a stable binary asteroid. Finally, applying our methodology to a monolithic body that is fractured by a series of impacts, we find that every time the body undergoes a fracture additional potential energy is liberated which may allow the body to be disrupted with a lower input of external energy, i.e., a slower spin rate.

2 Model

We define a rubble pile asteroid as a distribution of rigid bodies resting on each other, rotating in space, and in relative equilibrium. Note that they can be formed by fracturing an initially monolithic body or by reassembling a collection of rigid bodies from a catastrophic disruption. Due to the relatively small total mass of the system we can ignore failure limits due to high pressures, at most these would result in "softening" the contact points between the rigid bodies [3].

Consider a rubble pile composed of N rigid bodies resting on each other, denoted by $\mathcal{B}_i$, $i = 1, 2, \ldots, N$, with the entire set of bodies denoted as $\mathcal{B}$. We assume that the center of mass of the entire set is at the origin of the inertial frame, and that the system rotates about its maximum moment of inertia at a constant angular velocity $\boldsymbol{\Omega}$. Let the total angular momentum vector of the system define the inertial $\boldsymbol{z}$-axis and the $\mathcal{B}$ body-fixed $\boldsymbol{z}$-axis. Let the intermediate and minimum principal moments of inertia of $\mathcal{B}$ denote the body-fixed $\boldsymbol{x}$ and $\boldsymbol{y}$ axes. Each of the N components of this rubble pile have their own center of mass location relative to the origin, $\boldsymbol{R}_i$, and the attitude of each body relative to the body-fixed frame is denoted by a transformation matrix $\boldsymbol{T}_i$ which takes a vector expressed in the $\mathcal{B}$-fixed space into the $\mathcal{B}_i$ rigid body principal axis frame. Finally, the mass and inertia dyad of each body is denoted as M_i and $\boldsymbol{I}_i$, while the total mass and inertia dyad of the system is M and $\boldsymbol{I}$.

An important aspect of our research is how we choose to "collect" these N bodies into different sets. Specifically, we define collections of different rigid bodies resting on each other by capital indices I and J. Thus, body $\mathcal{B}_I$ consists of a set of bodies $i_1, i_2, \ldots$ that are resting on each other and, for the moment, are considered to define a rigid body separate from the rest of the collection. Depending on the relative resting geometry of the rigid bodies in our set, we can define a finite number of such "collections" and consider their properties in turn. We will generally just divide our asteroid into two collections, I and I', where I' are all the bodies not in I, so that these sets taken together define the entire asteroid, that they contain no common bodies, and that each of them consist of a connected set of rigid bodies, meaning that all the bodies in set I are in contact with each other and likewise for I'. Figure 1 shows all the ways in which a simple rubble pile can be partitioned into sets of connected rigid bodies.

An alternate way to think of our body is as an initially monolithic body that undergoes a series of fractures. Every fracture that cuts through the entire body creates additional component bodies that are resting on each other. Later we will see that such fracturing actually liberates potential energy that the system can use to subsequently evolve or, in extreme cases, even disrupt.

A final component of our model is the YORP effect, which changes the spin rates of bodies and their obliquity over time [9]. These changes can be systematic or may have oscillations, depending on the specific shape and initial spin state and obliquity of the asteroid [18]. Due to the YORP effect an asteroid will have a changing spin rate, leading to a changing angular momentum and energy. Even if we assume that the asteroid remains in principal axis rotation over time, as the angular momentum of the asteroid increases

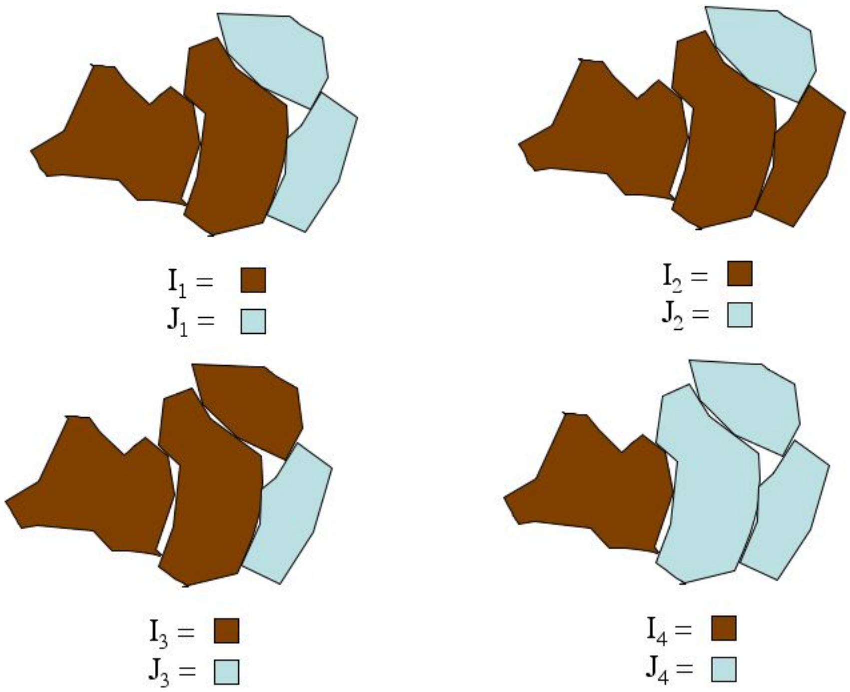

Figure 1: Example of all the possible connected partitions a particular rubble pile can be split into.

the system can transition from being in a minimum energy configuration to a non-minimum energy configuration [17]. Subsequent to this, the rubble pile is susceptible to reconfiguration. If the spin rate of the body continues to increase, components of the asteroid may undergo fission, meaning that they enter mutual orbit about each other.

3 Energy and Angular Momentum

To properly set up the discussion, the total energy and angular momentum of the rubble pile asteroid must be defined. In addition we define minimum energy configurations and the free energy of the system. The free energy is seen to control the evolution of the system and is a function of how the rubble pile is partitioned into different collections.

3.1 Total Energy and Angular Momentum

The rotating body has a total energy and angular momentum associated with it. We note that the total angular momentum is constant under any internal motions that the system goes through, the same is not true of the energy as this can be dissipated if there is sliding motion with friction between components. For the entire system the total kinetic energy, total potential energy, and total angular momentum are, respectively:

$$T = \frac{1}{2} I_z \Omega^2 \tag{1}$$

$$U = -\frac{\mathcal{G}}{2} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{dM(\boldsymbol{\rho}) dM(\boldsymbol{\rho}')}{|\boldsymbol{\rho} - \boldsymbol{\rho}'|} \tag{2}$$

$$\boldsymbol{H} = I_z \Omega \hat{\mathbf{z}} \tag{3}$$

where I_z is the maximum moment of inertia of the entire body, the potential energy U is the total self-potential of the body, and the total energy of the system is $E = T + U$. If we assume no relative motion between the rigid body components, then the self-potential is constant, and hence the kinetic energy of the system is constant and at a minimum for the current angular momentum, as we have assumed rotation about the maximum moment of inertia.

Now consider what happens when we no longer view the asteroid as a monolithic collection, but as a collection of rigid bodies that may move relative to each other. A first step along this path is to realize that the kinetic energy, potential energy and angular momentum of the asteroid can be partitioned into the sum of all of the energies and momentum of the individual bodies. In the extreme, we can view the entire system broken down into its constituent rigid bodies and find the quantities T, U and $\boldsymbol{H}$ which will be unchanged from the above in value[12]:

$$\begin{aligned} T = & \frac{1}{2} \sum_{i=1}^{N} \boldsymbol{\Omega} \cdot \boldsymbol{I}_i \cdot \boldsymbol{\Omega} \\ & + \frac{1}{2} \sum_{1 \leq i < j \leq N} \frac{M_i M_j}{M_i + M_j} (\boldsymbol{\Omega} \times \boldsymbol{R}_{ij}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{R}_{ij}) \end{aligned} \tag{4}$$

$$U = \sum_{i=1}^{N} U_{ii} + \sum_{1 \leq i < j \leq N} U_{ij} \tag{5}$$

where

$$U_{ij} = \begin{cases} -\frac{\mathcal{G}}{2}\int_{\mathcal{B}_i}\int_{\mathcal{B}_i}\frac{dM(\boldsymbol{\rho})dM(\boldsymbol{\rho}')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|} & i = j \\ -\mathcal{G}\int_{\mathcal{B}_i}\int_{\mathcal{B}_j}\frac{dM(\boldsymbol{\rho}_i)dM(\boldsymbol{\rho}_j)}{|\boldsymbol{R}_i-\boldsymbol{R}_j+\boldsymbol{\rho}_i-\boldsymbol{\rho}_j|} & i \neq j \end{cases} \tag{6}$$

It is convenient to also define the normalized potential energy $\tilde{U}_{ij}$ for later use:

$$U_{ij} = \mathcal{G}M_iM_j\tilde{U}_{ij} \tag{7}$$

Implicit in the mutual potentials U_{ij} are that they are also a function of the relative attitude between any two rigid bodies, $\boldsymbol{T}_{ij} = \boldsymbol{T}_j \cdot \boldsymbol{T}_i^T$ where $\boldsymbol{T}_{ij}$ represents a transformation from the frame fixed in $\mathcal{B}_i$ to the frame fixed in body $\mathcal{B}_j$. Likewise, the kinetic energy will change if the orientation of individual rigid bodies are shifted.

The Angular momentum is:

$$\boldsymbol{H} = \sum_{i=1}^{N}\left[\boldsymbol{I}_i \cdot \boldsymbol{\Omega} + M_i\boldsymbol{R}_i \times \boldsymbol{\Omega} \times \boldsymbol{R}_i\right] \tag{8}$$

$$= \sum_{i=1}^{N}\left[\boldsymbol{I}_i - M_i\tilde{\boldsymbol{R}}_i \cdot \tilde{\boldsymbol{R}}_i\right] \cdot \boldsymbol{\Omega} \tag{9}$$

where $\tilde{\boldsymbol{a}}$ represents the cross-product dyad formed from the vector $\boldsymbol{a}$, defined such that $\boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{a}} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \tilde{\boldsymbol{b}}$. Note that the quantity $\left[\boldsymbol{I}_i - M_i\tilde{\boldsymbol{R}}_i \cdot \tilde{\boldsymbol{R}}_i\right]$ represents the inertia dyad of $\mathcal{B}_i$ relative to the system center of mass.

To define physically meaningful decompositions of the rubble pile we split the current rubble pile into two collections of rigid bodies, denoted by I and J, and evaluate conditions for fission or movement of these two collection of bodies relative to each other. We assume in the following that $J = I'$, i.e., consists of all rigid bodies that are not in the set I, and that all of the rigid bodies in I and J are resting on each other, respectively. The total kinetic and potential energy and angular momentum of the system can be decomposed for two arbitrary collections into bodies $\mathcal{B}_I$ and $\mathcal{B}_J$ as [12]:

$$T = \frac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_I \cdot \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{I}_J \cdot \boldsymbol{\Omega} - \frac{1}{2}\frac{M_IM_J}{M_I + M_J}\boldsymbol{R}_{IJ} \cdot \tilde{\boldsymbol{\Omega}} \cdot \tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{R}_{IJ} \tag{10}$$

$$
\begin{aligned}
U &= U_{II} + U_{JJ} + U_{IJ} &\quad (11)\\
\boldsymbol{H} &= [\boldsymbol{I}_I + \boldsymbol{I}_J] \cdot \boldsymbol{\Omega} - \frac{M_I M_J}{M_I + M_J} \tilde{\boldsymbol{R}}_{IJ} \cdot \tilde{\boldsymbol{R}}_{IJ} \cdot \boldsymbol{\Omega} &\quad (12)
\end{aligned}
$$

It is usefull to further partition the total kinetic energy into the kinetic energy of each rotating body, $T_I = \frac{1}{2} \boldsymbol{\Omega} \cdot \boldsymbol{I}_I \cdot \boldsymbol{\Omega}$, and the kinetic energy due to mutual rotation of two bodies, $T_{IJ} = -\frac{1}{2} \frac{M_I M_J}{M_I + M_J} \boldsymbol{R}_{IJ} \cdot \tilde{\boldsymbol{\Omega}} \cdot \tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{R}_{IJ}$, leading to $T = T_I + T_J + T_{IJ}$.

3.2 Minimum Energy Configurations

Partitioning the system into these collections does not change the total energy or angular momentum of the system. There is a subtle difference, however, if we view the proposed collections $\mathcal{B}_I$ and $\mathcal{B}_J$ as individually rigid bodies. This means that, nominally, the self potentials U_{II} and U_{JJ} are constant but that the mutual potential energy U_{IJ} represents energy that can be used to transfer kinetic energy between rotational and translational motion of the system, leading to relative movement of the bodies.

If we consider this situation, we can define the energy of the system for different relative positions and attitudes of the two collections I and J. If we keep the total angular momentum constant, we can define the minimum energy configuration of the system for a given value of angular momentum. The variable here is the mutual position and attitude of the two bodies, $\boldsymbol{R}_{IJ}$ and $\boldsymbol{T}_{IJ}$. Once these are specified the resulting angular velocity vector can be solved for from:

$$
\boldsymbol{\Omega} = \left[\boldsymbol{I}_I + \boldsymbol{I}_J - \frac{M_I M_J}{M_I + M_J} \tilde{\boldsymbol{R}}_{IJ} \cdot \tilde{\boldsymbol{R}}_{IJ} \right]^{-1} \cdot \boldsymbol{H} \quad (13)
$$

For an arbitrary relative position or attitude, the resulting system may not be in a resting or orbital equilibrium, discussed later. However, out of all possible relative positions and poses there will exist a finite number that result in an equilibrium, and out of these at least one which is the overall minimum energy of the system, with the energy denoted as $E^m_{IJ}(H)$, where we note that this is a function of the angular momentum magnitude. This configuration is guaranteed to be a relative equilibrium, either resting or orbital, and defines the minimum energy for the partition I and J. Since bodies $\mathcal{B}_I$ and $\mathcal{B}_J$ may both consist of other rigid bodies resting on each other, there is no guarantee that the minimum energy for I and J is the overall minimum energy of the full collection. Thus, the configuration that gives the minimum energy for I and J may place some individual rigid bodies on either of these collections into a non-relative equilibrium, or moving one of these rigid bodies to the other set may yield a lower energy still.

Thus, for our original set of rigid bodies we can define the overall minimum energy configuration of the system as the set of positions and attitudes of each body that yields the overall minimum energy of the system. We note that the original configuration may not be the overall possible minimum energy configuration, as moving one rigid body to a different location, or placing it into an orbital equilibrium, may result in a lower total energy. If we open up our system to these further reconfigurations we can define the absolute minimum energy configuration for the entire system, $E^m(H)$, to be the configuration and placement of all the rigid bodies which minimizes the energy at a given angular momentum. By definition we have $E^m(H) \leq E^m_{IJJ}(H)$. Since all are a function of the angular momentum, as the angular momentum of the system changes the minimum energy configuration of the system may likewise shift [17]. Once a system is no longer in its minimum energy configuration, an external disturbance can then cause the system to change its configuration and seek out a lower energy state.

3.3 Free Energy

A useful concept in understanding orbital movement of two partitions relative to each other and its implications is the free energy [13]. The free energy is simply defined as the total energy minus the self-potential energies of the collections being considered. Thus for a rubble pile decomposed into a set I and J, the free energy of the system is:

$$\begin{aligned} E^F_{IJ} &= E - U_{II} - U_{JJ} && (14) \\ &= T_I + T_J + T_{IJ} + U_{IJ} && (15) \end{aligned}$$

The free energy only includes the energies of the system that are free to change under mutual exchange of forces and moments, and assumes for the moment that the rigid body collections I and J are fixed relative to themselves.

As the potential energy is always negative, we note that $E^F_{IJ} \geq E$, and that for the original body treated as a monolith the free energy of the system is $E^F = E - U = T$ and is a maximum. As different partitions are considered, the free energy will decrease from E^F, leading to the compound inequality:

$$E^F \geq E^F_{IJ} \geq E \quad (16)$$

For a given rotation rate (angular momentum), the free energy is a function over the partition I and J only and that the possible values of free energy expand as we consider finer partitions of the system.

Given the free energy of a system comprised of two partitions in orbit about each other, we can define strict constraints on possible final outcomes for the system [12]. Specifically, as the system evolves the potential and kinetic energy of the system can be exchanged. If the free energy is positive, we find that the system can escape, leading to $T_{min} = E^F_{IJ}$ and $U_{IJ,min} = 0$. If the free energy is negative, $E^F_{IJ} < 0$, we know that $T_{min} = 0$ and $U_{IJ,min} = E^F_{IJ}$, which implies that there is a maximum distance between the two components that the system can achieve. We also know, trivially, that $E^m_{IJ} \leq E^F_{IJ}$, where E^m_{IJ} is the minimum energy of the I and J partition of the system at a fixed value of angular momentum.

For a monolithic body or the complete collection of components in rotating equilibrium the free energy equals the kinetic energy, and conservation dictates that the kinetic energy cannot change – it cannot increase or decrease as there is no other "place" for the energy to go. When the free energy is decreased by a partition of the bodies into two sets, and the subsequent introduction of U_{IJ}, then there is the possibility for changes in the kinetic energy to be absorbed or liberated by the mutual potential. These changes come about by shifts in position or in attitude between the two collections, all the while conserving angular momentum. We note that if the entire configuration is at a minimum energy configuration (for a given angular momentum), then there are no possible reconfigurations of the system that are allowed [1]. Specification of these minimum energy configurations are of interest, as they rigorously define the amount of "movement" possible in the system at a given energy above this minimum.

4 Equilibria, Reconfigurations, Fission and Stability

To capture conditions for rigid body collections to shift relative to each other requires that the forces and moments be defined and computed.

4.1 Relative Forces and Moments

The force and moment between two bodies, or partitions, is represented as the gradient of the potential with respect to the mutual position $\boldsymbol{R}_{ij} = \boldsymbol{R}_j - \boldsymbol{R}_i$ or the mutual attitude $\boldsymbol{\theta}_{ij}$, respectively. The vector $\boldsymbol{\theta}$ represents the axis-angle attitude representation and denotes an arbitrary variation of the mutual attitude of the two bodies. The indices i and j can represent single bodies or, as forces and moments add linearly, can represent collections of bodies. We denote these as:

$$\boldsymbol{f}_{ij} = -\frac{\partial U_{ij}}{\partial \boldsymbol{R}_{ij}} \tag{17}$$

$$\boldsymbol{m}_{ij} = -\frac{\partial U_{ij}}{\partial \boldsymbol{\theta}_{ij}} \tag{18}$$

and note the basic results, $\boldsymbol{f}_{ij} + \boldsymbol{f}_{ji} = \boldsymbol{0} = \boldsymbol{m}_{ij} + \boldsymbol{m}_{ji}$. Thus, when viewed as a closed system, there is no net force or moment arising from internal gravitational attractions.

In addition to the gravitational forces and moments acting on the bodies, there will also be a relative centripetal acceleration acting between any two rigid bodies or partitions, due solely to the fact that the rigid bodies are spinning, and with the explicit form:

$$\boldsymbol{a}_{ij} = -\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{R}_{ij} \tag{19}$$

where again $\boldsymbol{a}_{ji} = -\boldsymbol{a}_{ij}$. From D'Alembert's principle this can be viewed as a centrifugal force acting on each of the bodies if multiplied by the reduced mass $M_i M_j/(M_i + M_j)$. The centripetal acceleration does not induce a relative moment between the two bodies.

For the resultant force we sum the gravitational and centrifugal to define:

$$\boldsymbol{F}_{ij} = \boldsymbol{f}_{ij} + \frac{M_i M_j}{M_i + M_j}\boldsymbol{a}_{ij} \tag{20}$$

The force we state is generally assumed to be applied to the center of mass of the body in question, although the force for an arbitrary mass distribution will in general not lie along the relative position vector $\boldsymbol{R}_{ij}$ (see Fig. 2). The actual application point of a force on a rigid body is not unique, and for a total force acting on the center of mass of the body j, the actual application point can be at any point along the line parallel to $\boldsymbol{F}_{ij}$ and passing through the center of mass of j. Conversely, the total moment acting on the body is not associated with any given point, but is a function of where the force is assumed to act.

Using these realizations, it is possible to define a resultant force and moment acting on a given rigid body. Displacing the force in a direction perpendicular to $\boldsymbol{F}_{ij}$ is equivalent to applying an additional moment to the body. For any given force and moment pair it is always possible to define the unique "wrench" for that body [4], which is defined by a unique offset distance $\boldsymbol{d}_{ij}$ from the center of mass along which the original force $\mathbf{F}_{ij}$ acts, which induces a moment $-\boldsymbol{d}_{ij} \times \boldsymbol{F}_{ij}$ such that the total moment acting on the rigid body $\boldsymbol{m}_{ij} - \boldsymbol{d}_{ij} \times \boldsymbol{F}_{ij}$ only has a component along the direction $\boldsymbol{F}_{ij}$. This unique offset distance can be defined as:

$$\boldsymbol{d}_{ij} = \frac{\boldsymbol{F}_{ij} \times \boldsymbol{m}_{ij}}{\boldsymbol{F}_{ij} \cdot \boldsymbol{F}_{ij}} \tag{21}$$

The wrench defines a unique line of action for the total gravitational force acting on the body, with the only remaining moment acting parallel to this line of action, inducing a twisting moment about the wrench line. Given the symmetry of two rigid bodies relative to each other, we note that the wrench offset of both bodies is identical, or that $\boldsymbol{d}_{ij} = \boldsymbol{d}_{ji}$. Thus, if the forces are initially collinear along the line connecting the centers of mass (which they are not in general), they will remain collinear when transported to the wrench line. Figure 2 depicts the total moment and force acting about the center of mass and the equivalent wrench.

The contact forces that act between two rigid bodies arises from the frictional model between the two bodies. For simplicity we assume a Columb friction model, thus the maximum lateral force that a common surface can sustain before slipping is μN, where N is the normal force between the contact surfaces and μ is the coefficient of friction. Similarly, the moment that can be resisted before slipping occurs equals the integral of $\frac{\mu}{A}\boldsymbol{\rho} \times \boldsymbol{N}$ over the contact surface, where $\boldsymbol{\rho}$ is measured from the rotation point, nominally where the wrench intersects the surface, and A is the area of the surface. More sophisticated formulations are possible.

Sliding and slipping motion will be very important for the migration of regolith up to the size of boulders and blocks on the surfaces of asteroids [7]. For larger collections of blocks and bodies we propose a strong surface roughness assumption that the finer scale structure is rough enough so that $\mu = \infty$ can be used. This limits the types of motion we need to consider to

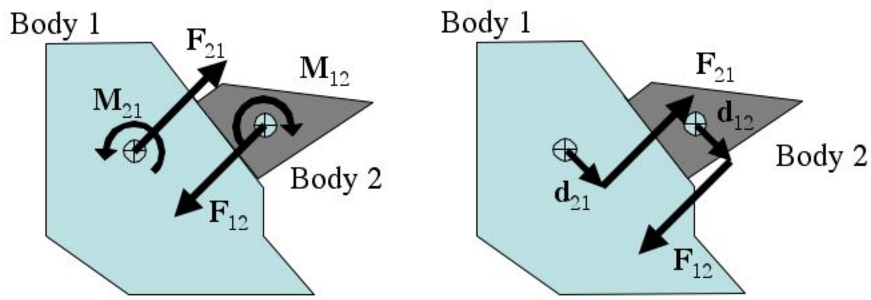

Figure 2: Relative forces, moments and wrenches acting on two rigid bodies. Since the wrench of Body 2 lies outside of its constraint surface, it will tip over.

tipping, rolling without slipping, and fission. The rationale for this model is that the surface of two rubble piles resting on each other will have so many non-convex intrusions into each other that lateral or sliding motion across each other cannot occur without component bodies breaking, a phenomenon we do not consider here, but which is of interest.

4.2 Resting Equilibrium and Stability

For a given rubble pile asteroid, as its spin rate increases over time, either gradually or abruptly, its components may undergo shifts as the total centripetal acceleration increases. For bodies balanced at a single point, such as spheres and ellipsoids resting on each other (studied in [17]), these transitions occur when the current configuration is no longer the minimum energy configuration. Once such a transition is passed the bodies will ideally roll across each other to a new configuration in the presence of disturbances. In the absence of energy dissipation, the system would rock back and forth about the new relative equilibrium, however when dissipation is taken into account the system should eventually settle into its new, minimum energy configuration.

For non-smooth rigid blocks resting on each other the situation is more complex. For two bodies or collections resting on each other the condition for one to tip relative to the other is for the total wrench of either body to lie outside of the contact region between the bodies, or for forces lateral to the contact surfaces to overcome the Columb friction force. For a given rigid body with a net force and moment acting on it, it is simple to compute the total wrench and compare it to the contact surface. For a given body and a given partition, we can uniquely define the spin rate at which such a shift will occur. These shift spin rates can be ordered in terms of magnitude and compared with the current spin rate.

To compute this one must define the "Intersection Cone" for the two bodies relative to each other. This is the set of rays from a given location in one rigid body to every contact point between the two bodies. Thus, this intersection cone is defined as a function of the rigid bodies or partitions, I and J, and as a function of position, and is denoted as $\mathcal{C}_{IJ}(\boldsymbol{r})$. To determine whether one rigid body I will tip relative to J, take the position to be the center of mass of I, $\boldsymbol{r}_I$, plus the wrench offset, $\boldsymbol{d}_{IJ}$, and compare the intersection cone to this point with the wrench line direction $\boldsymbol{F}_{IJ}$. The tipping spin rate is then defined as the spin rate when the wrench line lies on or outside of the intersection cone, and is defined as Ω^T_{IJ}. This definition assumes that the contact points and surfaces between the two bodies define locally convex regions, i.e., that the two bodies do not have interlocking pieces. We will make the same assumption later for our fission computations. For example, Body 2 in Fig. 3 would tip over as its wrench lies outside of its constraint surface, i.e., the constraint surface can supply no restoring moment to keep the body in equilibrium. However, Body 1 does not satisfy the tipping condition, pointing out that tipping is not a symmetric condition, i.e., only one body may satisfy it. That being said, once either body starts to tip both bodies are affected due to mutual gravity and surface forces. A similar definition can be used to define the spin rate at which two collections will slide relative to each other, Ω^S_{IJ}, although the computation of this limit is more difficult as it involves computing the normal force between the bodies at each contact point and surface.

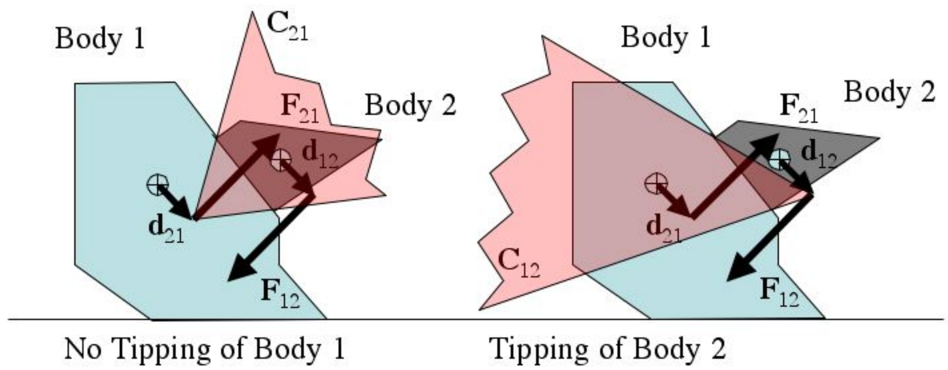

Figure 3: Constraint cones and wrenches for the two bodies. Note that Body 1 does not satisfy the tipping condition, but that Body 2 does as its wrench lies outside of its constraint cone.

These spin rate limits can be defined for every possible partition of the system, once computed we find the minimum spin rate for such shifting or tipping to occur, defined as:

$$\Omega^T = \min_{I,J} \Omega^T_{IJ} \tag{22}$$

and similarly for Ω^S. Then, as the total spin increases, once this minimum spin rate is reached the system will evolve dynamically. Detailed motion of the tipping body and its subsequent interactions with the entire body will be complex, require numerical simulation and choice of interaction models, and include the possibility of further collapse of partitions. Such an event can occur for an isolated rigid body or can initiate a global "landslide" on an asteroid. In the aftermath of such a landslide, when all the blocks have settled again into a resting equilibrium, an entirely new list of tipping spin rates must be determined. Note that angular momentum will be conserved across such an evolution, but that energy will not be. Further, the reconstituted system will likely be in non-principal axis rotation and every point in the body will be subject to a periodically varying acceleration. Relevant questions for this are how long it takes for the body to relax into principal axis rotation again. Also, as the YORP effect is sensitive to small changes in a body's geometry, the YORP effect may change significantly, as has been hypothesized for the asteroid Itokawa [16].

4.3 Fission Conditions

For all configurations and partitions, there exists a spin rate that results in the two components entering orbit about each other, instead of just tipping relative to each other. This is defined as the fission limit and it will again be a function of how the rigid body collections are partitioned. For bodies in resting equilibrium at a single contact point, this condition occurs when the total force between the components goes to zero [17]. For two rigid bodies resting on each other this occurs when the force across their contact surface goes to zero or switches sign at every point of contact of the two bodies. Once this occurs the bodies can "float" apart from each other and in general will enter orbit about each other.

The total force between partitions I and J is:

$$\boldsymbol{F}_{IJ} = -\frac{M_I M_J}{M_I + M_J}\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{R}_{IJ} - \frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \tag{23}$$

If the contact surfaces between the bodies allows unfettered displacement along the lines connecting the centers of mass (our previous convexity assumption), a condition for fission is then $\boldsymbol{F}_{IJ} \cdot \boldsymbol{R}_{IJ} \geq 0$, which can be rewritten as:

$$-\frac{M_I M_J}{M_I + M_J}\boldsymbol{R}_{IJ} \cdot \tilde{\boldsymbol{\Omega}} \cdot \tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{R}_{IJ} - \frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \geq 0 \tag{24}$$

or

$$2T_{IJ} - \frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \geq 0 \tag{25}$$

This condition can be normalized by dividing through by the effective mass of the partition, $M_I M_J/(M_I + M_J)$ to find:

$$-\boldsymbol{R}_{IJ} \cdot \tilde{\boldsymbol{\Omega}} \cdot \tilde{\boldsymbol{\Omega}} \cdot \boldsymbol{R}_{IJ} - \mathcal{G}(M_I + M_J)\frac{\partial \tilde{U}_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \geq 0 \tag{26}$$

where $(M_I + M_J) = M$ is the total mass of the asteroid, independent of the partitions I and J. Thus, for a given configuration the fission condition is only a function of the total spin rate, Ω, and thus define the fission spin rate for a partition I and J as the spin rate Ω_{IJ}^F such that $\boldsymbol{F}_{IJ} \cdot \boldsymbol{R}_{IJ} = 0$. For our assumed fission condition we can solve for this fission rate:

$$(\Omega_{IJ}^F)^2 \quad = \quad \frac{\mathcal{G}M}{-\boldsymbol{R}_{IJ} \cdot \tilde{\hat{\boldsymbol{z}}} \cdot \tilde{\hat{\boldsymbol{z}}} \cdot \boldsymbol{R}_{IJ}} \frac{\partial \tilde{U}_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \tag{27}$$

Associated with this fission spin rate is a corresponding angular momentum and free energy. The fission condition is independent of the rotational kinetic energy of the two components, T_I and T_J and instead just depends on the total force between the two components being repulsive due to a large enough centripetal term (see Fig. 4).

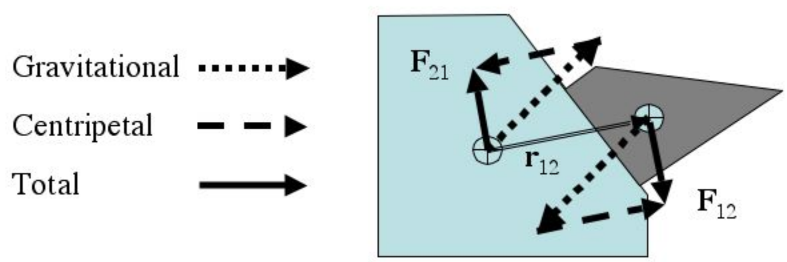

Figure 4: Fission conditions. Once the net force acting on each body is repulsive, the two bodies are free to enter orbit.

For the classical N-body problem Euler's Theorem on homogeneous functions tells us that $-\partial U_{IJ}/\partial \boldsymbol{R}_{IJ} \cdot \boldsymbol{R}_{IJ} = U_{IJ}$ [8]. This theorem only holds when the system is comprised of point masses with unconstrained degrees of freedom. For a system of rigid bodies, or even point masses with constraints, this theorem no longer holds and is replaced by a weaker condition [12]:

$$-\frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \quad > \quad 2U_{IJ} \tag{28}$$

This result is derived in [12] under a restriction on the distance between the bodies that, for our systems with two bodies in contact, is generally violated. However, the proof in [12] also provides indication that this restriction may be relaxed. What we hypothesize here, for future investigation, is that the above inequality holds for resting bodies. We further hypothesize that a general proportionality exists independent of, or weakly dependent on, the partition I and J:

$$-\frac{\partial U_{IJ}}{\partial \boldsymbol{R}_{IJ}} \cdot \boldsymbol{R}_{IJ} \quad \propto \quad U_{IJ} \tag{29}$$

Thus, the fission condition is reduced to:

$$T_{IJ} + \alpha \tilde{U}_{IJ} \quad \geq \quad 0 \tag{30}$$

where α is a proportionality constant less than unity given Inequality 28. If this hypothesis is true, then the minimum fission rate of a given system will correspond to a combination of both the maximum normalized mutual potential of the system, $\tilde{U}_{IJ}$, and the maximum distance between the two mass collections, $\boldsymbol{R}_{IJ}$, measured perpendicular to the rotation axis. Generally, the mutual potential scales as $\tilde{U}_{IJ} \propto -\frac{1}{|\boldsymbol{R}_{IJ}|}$ and thus the maximum distance between the centers of mass generally corresponds to the maximum mutual potential energy between the components.

From this line of argument and hypothesis we find that $\Omega^F_{IJ} \propto 1/R^{3/2}_{IJ}$. Thus, the first components of the system to fission will be those with the largest separation between their mass centers (perpendicular to the rotation axis). Simple examples show that this is true for ideal shapes such as spheres and ellipsoids resting on each other. Specifically, two equal size spheres resting on each other will fission at half the spin rate of a small particle resting on a larger sphere (the usual surface disruption spin rate). Also, a sphere and ellipsoid resting on each other with similar masses can fission at spin rates up to three times as slow as the classical surface disruption spin rate of a sphere. The results for sphere-ellipsoid systems are detailed in [17], further investigation of these effects for collections of rigid bodies is of interest.

4.4 Stability and Free Energy

Once fission occurs, the stability of the system dynamics takes over. The orbital and rotational evolution of the system is then free of constraints other than conservation of angular momentum and conservation of free energy, and follows the dynamics of the "Full 2-body Problem" [12, 11]. We note that there are strict constraints which can be placed on the system, including necessary conditions for mutual escape of the bodies and sufficient conditions for them to not impact in the future [12]. In terms of system stability, we note three possible outcomes of fission for a particular partition IJ, assuming that each body $\mathcal{B}_I$ and $\mathcal{B}_J$ remains intact without any internal shifting or tipping:

1. $E^F_{IJ} > 0$: In this case the system has a positive free energy and a mutual escape of the two partitions is possible and in general likely. Should this occur, the body has undergone a catastrophic disruption. Specific limits on mass distribution and spin rates for this to occur have been defined previously in [17] for sphere-ellipsoid systems, but have yet to be studied in detail for non-smooth bodies. A value of $E^F = 0^+$ corresponds to the true minimum energy for a given rubble pile to undergo catastrophic disruption.

2. $0 > E^F_{IJ} > E^m_{IJ}$: In this case the system has a negative free energy and cannot undergo a catastrophic disruption under internal dynamics alone. It is also not at its minimum energy configuration, however, and thus may be able to undergo significant orbital and attitudinal evolution. Outcomes include re-impact, with an attendant loss of energy and re-distribution of the system's rigid bodies, or energy dissipation in orbit until the system approaches the minimum energy configuration. It is important to note that a binary system in orbit about the sun may still experience disruption even if its free energy is negative, as once the system achieves a large enough distance between components the solar tide can add to the free energy and allow the system to disrupt. Of course, the solar tide may also decrease the angular momentum, allowing a reimpact of the two components.

3. $E^F_{IJ} = E^m_{IJ}$: In this case, which in general will only happen for the ideal case of a single point of contact between the bodies (in addition to the proper mass distribution), the system immediately enters a stable relative equilibrium. If impact between the fissioned bodies does not

occur, and if the subsequent tidal evolution of the bodies does not cause further reconfiguration of either of the partitions, then this should be the eventual end state of a fissioned system with negative free energy. It is also possible for a system with an initially positive free energy to end up in this situation, although excessive amounts of energy must be dissipated fast enough to prevent the system from mutual escape [1].

The fission spin rate decreases as the distance between the centers of mass between two collections increases, independent of the mass distribution between these two collections. It is instructive to understand the relation between this geometric effect and the resulting free energy of the system. Rewrite the free energy, extracting the effective mass $M_I M_J/(M_I + M_J)$ from the mutual kinetic energy and potential:

$$E^F_{IJ} = T_I + T_J + \frac{M_I M_J}{M_I + M_J}\left[-\frac{1}{2}\boldsymbol{\Omega}\cdot\tilde{\boldsymbol{R}}_{IJ}\cdot\tilde{\boldsymbol{R}}_{IJ}\cdot\boldsymbol{\Omega} + \mathcal{G}M\tilde{U}_{IJ}\right] \quad (31)$$

The term $T_{IJ}+U_{IJ}$ is not necessarily positive when fission occurs. However if this term is positive, then the free energy is positive from Inequality 28. If this term is not positive, it is still possible for the free energy to be positive based on contributions from T_I and T_J. Now we note that the mass distribution between the two components plays an important role for the free energy. Define the mass fraction of the system by $\nu = \frac{M_I}{M_I+M_J}$, thus the effective mass equals $\nu(1-\nu)(M_I + M_J) = \nu(1-\nu)M$. Also, we can define the mass normalized inertia dyads for each body as $\bar{\boldsymbol{I}}_i = \boldsymbol{I}_i/M_i$, these being a function of the geometric distribution of the mass alone. Then it is meaningful to extract the effective mass from the rotational kinetic energy terms, leading to a mass normalized expression for the free energy:

$$\bar{E}^F_{IJ} = \frac{1}{2}\boldsymbol{\Omega}\cdot\left[\frac{\bar{\boldsymbol{I}}_I}{1-\nu} + \frac{\bar{\boldsymbol{I}}_J}{\nu}\right]\cdot\boldsymbol{\Omega} - \frac{1}{2}\boldsymbol{\Omega}\cdot\tilde{\boldsymbol{R}}_{IJ}\cdot\tilde{\boldsymbol{R}}_{IJ}\cdot\boldsymbol{\Omega} + \mathcal{G}M\tilde{U}_{IJ} \quad (32)$$

In this form, we note that if one of the mass fractions is small, but has a non-vanishing moment of inertia, that the free energy of the system becomes positive. This is the situation for a small piece of regolith resting on the surface of a monolith. If the body spins rapidly enough for the regolith to enter orbit, then it becomes possible for the regolith to escape the body (exceptions occur when the mass distribution is rotationally symmetric or spherical, which are degenerate cases). If the mass fractions are more equal to each other the free energy is minimized, meaning that a negative free energy is more likely. The interplay between the mass fraction and the separations of the mass centers is complex and geometry dependent.

These issues are explored in [17] for contact binary asteroids consisting of a sphere and ellipsoid resting on each other. For the sphere-ellipsoid system the moment of inertia of the sphere can be neglected, as there are no gravitational torques which can be placed across the body after fission. For collections of rigid bodies this is not a good assumption, and implies that realistic rubble pile asteroids are more likely to have positive free energy and hence are more likely to disrupt when spun to fission.

Additional results for the sphere-ellipsoid system include conditions for the reconfiguration of this class of contact binary asteroids and the relation between fission rate, mass fraction, and the stability of the fissioned system. Starting from the conditions stated here, a similar analysis can be performed for classes of non-smooth rigid bodies resting on each other.

If the system has entered a stable binary configuration it is still possible for it to undergo continued evolution, as the YORP and BYORP effects may cause further expansion or contraction of the system. The YORP effect can continue to drive this system if the primary body is still subject to continued rotational acceleration. If it is driven to fission material again, the excess angular momentum it sheds by sending components into orbit can be transferred direction to the existing orbit and synchronous rotation state of the secondary, causing the system to further expand [14]. The BYORP effect accounts for solar radiation pressure acting differentially on the secondary, which can couple with the orbital equations and cause the system to expand or contract [2]. The BYORP effect has been hypothesized but has not been verified as of yet. These evolutionary considerations take over after formation of a stable binary.

5 Discussion

In this paper we have outlined some basic principles for how a rubble pile asteroid's configuration can change as its spin rate changes. From our analysis we can also define the spin rate at which the collection can fission into two disjoint pieces, and an explicit condition to check for which of the possible collections of rubble will undergo this fission. Once this fission rate is reached, we also provide a condition for whether it is possible for the system to undergo a catastrophic disruption. This is seen to be wholly dependent on the mass distribution of the rubble pile components.

Thus, given a collection of rigid bodies resting on each other, we can in principle compute whether this collection can have its spin rate increased to fission without undergoing a reconfiguration. If so, we can also determine whether the fissioned system is stable and will remain bound to each other or is unstable and can mutually escape. Our conditions are simplified in the sense that we only consider the mutual orbit and translational dynamics of two collections of rigid bodies relative to each other. Smaller regolith or boulders that may separate from either collection are themselves subject to a system where they can be easily ejected. Such ejections only represent a small fraction of the mass, however. Thus our criterion considers the more interesting case where a sizable fraction of the body is lost.

One interesting result is that a rubble pile asteroid can be given sufficient energy to catastrophically disrupt without a large impact or other temporally focused energy input. Instead, a subtle effect such as YORP can spin the body to a rate where it will naturally separate and undergo mutual escape. We note that whether or not such a body will escape once it fissions depends on its free energy, which depends on how the mass is distributed within the different rigid body components that make up the system. In other words, the component shapes of the rigid body both control its fission spin rate and its fate when spun to disruption. Thus, given a distribution of rigid bodies resting on each other, it becomes possible to determine its fate. Such an analysis has been applied to the asteroid Itokawa [15], modeling its overall shape morphology as two ellipsoids resting on each other. In that case it was found that the body would be susceptible to fission at a spin period of only 6 hours but that the resulting system would have a negative free energy and thus could not mutually escape. Thus, it is reasonable that we find the Itokawa system to be a contact binary system.

This also provides insight into how the fracturing of a monolithic body changes the energy of a system. Given an initially competent rock, one can imagine a series of impacts large enough to fracture the body but small enough to not disrupt it or change its relative positions and orientations. For the initial monolithic body we note that its free energy, E^F is positive but that its mutual potential energy is zero, all the potential energy being caught up in the self potential of the body. If the body is then fractured into two pieces, 1 and 2, the free energy decreases by the new mutual potential U_{12}. The fission rate of the system is now defined as Ω_{12}^F, and in general will be proportional to $1/R_{12}^{3/2}$. Another quantity of interest is the free energy of the system at this fission rate, $E_{12}^F(\Omega_{12}^F)$. Whether this is positive or negative controls if catastrophic disruption may occur due to an increase in spin rate alone. If the body is further fragmented, the original partition into bodies 1 and 2 is still viable, however there are other possible partitions that may have a lower fission spin rate. In general, further fragmentation will only decrease the minimum fission spin rate for the body and allow for larger values of free energy, and thus make it easier to disrupt the asteroid. In this sense, a fragmented asteroid is easier to disrupt than a non-fragmented body.

To provide additional data to understand these issues it is necessary to obtain higher resolution imagery of asteroids and study their morphology, including size distribution of competent blocks, global shapes and estimates of their YORP coefficients to understand their recent past. Such studies can be partially determined by radar observations, and are best performed by rendezvous missions.

6 Conclusions

This paper states the general conditions for a rubble pile asteroid composed of rigid bodies resting on each other to undergo reconfiguration events and fission. The important quantities for the system are stated as a function of the different ways in which the rubble pile components can move relative to each other. The paper bypasses the difficult question of modeling the specific dynamics of the asteroid following a reconfiguration event. For fission events the free energy of the system does place strict constraints on what possible final outcomes for the system may be. These include catastrophic disruption of the system under its internal dynamics alone. This is significant as it implies that a subtle effect such as YORP can spin a rubble pile asteroid at a sufficient rate for the body to catastrophically disrupt, with no additional energy applied. Furthermore, the rotation rates for this disruption may be significantly less than the spin rate for surface particles to enter orbit about the asteroid. These fission rates are generally controlled by the largest separation between mass centers of the asteroid, determination of these minimum rotation rates requires that all different possible configurations of the given rubble pile be considered. Given the minimum fission rotation rate the free energy can be computed and predictions about the future evolution of the disrupted body can be made.

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