Lecture notes on general relativity

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Foreword

This document contains lecture notes on the course of general relativity FX2 H97 read in the fall semester 1997 at the Physics Institute of NTNU. It is not aimed to be a textbook, but rather a guide through the textbooks and other literature recommended to the students.

The basic textbook for this course is Part I of "General relativity" by R. M. Wald, [6].

It is highly recommended to the students to solve the exercises contained in the lecture notes, because some of them illustrate important applications of the main text.[1] Similar exercises (or some other, which can be found e.g. in [4] or [6]) will be used also at the examinations.

[1]There exist also solutions of the exercices, which can be requested at the author.

Chapter 1

Introduction

Classical non-relativistic physics is supposed to be the theory describing behaviour of particles, extended bodies or physical fields. It is assumed that the play of these physical entities takes place on the stage of the physical space with fixed properties well-described by Euclidean geometry and that it goes on in time, which is an even simpler one-dimensional space. Special relativity disclosed that the perception of space and time as different quantities is in fact an illusion dependent on the dynamical status of the observer, and that these quantities are actually only different projections of one entity – the spacetime – which has Minkowskian geometry. The subject of relativistic physics is thus the reformulation of classical physics in view of this fact, making its laws more symmetrical, i.e., simpler for those who are accustomed with this view. Relativity revealed that introducing gravity into its theory requires that one abandons the assumption of flat Euclidean space (or spacetime) and to accept its (pseudo-) Riemannian geometry. The geometry of the spacetime is thus no longer a background for physics, but one of the physical fields (i.e., the gravitational field) mutually interacting with others.

This demands more imagination to understand general relativity and more powerful mathematical tools to handle its calculations. The latter, which is first of all the differential geometry of pseudo-Riemannian manifolds, can be treated on different levels. The classical approach used by Einstein is based on tensor analysis using a notation with indices corresponding to components in a coordinate basis. This formalism is used in many textbooks, e.g., in [5] to which we will also refer or in [7]. More advanced formalisms using tetrads and differential forms have been developed to save at least a part of the tedious computations of spacetime geometry and to yield directly the view of local observers. This approach is used in 'the relativistic bible' – MTW [4] (which is, however, very extensive, so that it is better used as a supplementary source rather than a basic textbook) and in the textbook by Wald [6], which will be used as the basic one in this course. This approach is sometimes referred to as too formal to the detriment of the physical insight to the theory. However, an investment of effort to learn a more elegant form of mathematics can not only simplify tedious calculations but also facilitate the understanding of the physics. Certainly, the same help is yielded by physical intuition. It is thus a matter of individual choice which one of these parallel ways will be preferred. In any case, the underlying mathematics is only a tool, which will be continuously developed to handle the recognized physical nature. Nowadays a relativistic physicist should understand both approaches to be able to read the literature. To facilitate this understanding, a brief summary of differential geometry is added here in an appendix.